



Finite-Time Passivity for Atangana–Baleanu–Caputo Fractional-Order Systems with Nonlinear Perturbations

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Abstract

This paper investigates the problem of finite-time passivity for a class of nonlinear Atangana–Baleanu–Caputo (ABC) fractional-order systems (FOs). Firstly, a new definition of finite-time passivity for the FO nonlinear systems in the frame of the Atangana–Baleanu derivative of Caputo type is introduced. Then several sufficient conditions in the form of linear matrix inequalities (LMIs) are presented to guarantee that such the system is robustly finite-time passive. Finally, two numerical examples are given to verify the theoretical results.

Keywords Finite-time passivity · Atangana–Baleanu–Caputo fractional-order systems · Finite-time boundedness · Linear matrix inequalities

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1 Introduction

In recent years, many researchers have introduced some new fractional operators with different kernels to better describe reality phenomena. For example, a generalized fractional-order derivative that covers Caputo derivative as a simple case was introduced and analyzed in [27]. Khalil et al. [28] developed a new definition of local fractional derivative which is called the conformable fractional derivative. CF fractional derivative with an exponential kernel was given by Caputo and Fabrizio [8]. In 2016, Atangana and Baleanu [3] presented a new fractional operator with a nonlocal and nonsingular kernel called Atangana–Baleanu (AB) fractional derivative by the generalized Mittag–Leffler function. As noted by many scholars, AB fractional derivatives have been successfully applied in various fields, for instance, chaos [4], circuit model [2], image processing [17], electromechanical model [1], HIV/AIDS epidemic model [39], cancer treatment mathematical model [31, 37], hepatitis B virus model [44], power system model [43], and so on. Therefore, AB fractional order has received considerable attention [5]. Several interesting works on some problems about the existence of local, extremal, and global solution [25], finite-time stability [36], optimal control [23, 38], numerical simulations [15, 24], and Lyapunov stability [30, 34] have been reported in the literature.

In practical application, it always needs to concern the behavior of the FOs over a finite time [26, 47]. Therefore, the concept of finite-time stability (FTS) and finite-time boundedness (FTB) was addressed for fractional-order systems. Many interesting results on FTS and control problems for fractional-order systems in sense of the Caputo type have been reported in recent years [9, 13, 14, 20, 29, 42]. The problem of FTS for linear Caputo–Katugampola FOs with a constant time delay was first considered by Ben Makhlouf and Nagy [7]. The results in [7] were extended for Caputo–Katugampola fractional-order neural networks in [21]. Several sufficient conditions were derived in [21] to ensure the existence of a unique equilibrium point and the equilibrium point is finite-time stable. In addition, the problem of FTS for a class of linear conformable FOs was studied in paper [6]. In this works, several FTS and FTB conditions in terms of matrix inequalities were presented based on using some properties of conformable fractional calculus. Very recent year, the authors in [36] investigated the problems of FTS and FTB for linear ABC fractional-order systems. By using the Laplace transform of ABC derivative combined with the finite-time stability theory, some stability and stabilizability conditions were derived in [36] for the linear considered systems.

The passivity theory intimately related to circuit analysis methods has received wide attention from the control community since the 1970s. The main advantage of this theory is that the passive properties of a system can keep a system internally stable. Therefore, the passivity theory is one of the effective tools to study the stability analysis of dynamic systems with nonlinear perturbation. The concept of passivity has been extensively investigated for both integer-order systems and fractional-order one in the context of Lyapunov stability [16, 33, 35]. However, in some practical process, the main attention may be related to the behavior of the dynamical systems over a fixed finite-time interval. Therefore, it is vital to study the passivity in the context of finite-

time stability theory. By introducing the concept of passivity to finite-time analysis theory, the concept of finite-time passivity has been explored. The dynamical systems are said to be finite-time passive if the systems are finite-time stable and satisfy the given passive index. Based on this idea, some good theoretical results on the problem of finite-time passivity for some kind of dynamical systems such as switched nonlinear systems [45], coupled reaction-diffusion neural networks with multiple weights [46], delayed neural networks [32], stochastic Markov jump neural networks [10], and Caputo fractional-order neural networks [41] have been published. It should be noted that almost all of the current results on the problem are focused on integer-order dynamical systems, and few works are devoted to Caputo fractional-order systems [41], not to mention ABC fractional-order systems. To the best of our knowledge, there is not any work on the finite-time passivity of nonlinear ABC fractional-order systems, which motivates the present study.

Motivated by the above discussions, in this paper, we investigate the problem of finite-time passivity for nonlinear Atangana–Baleanu–Caputo fractional-order systems. Our major contributions are reflected in the following aspects:

- (i) By using the finite-time stability theory and some properties of Atangana–Baleanu calculus, we present an improved condition for the problem of FTB for nonlinear ABC fractional-order systems. In comparison with the existing results, the considered model in this paper is more general than the model in [36].
- (ii) The definition of finite-time passivity is extended to ABC fractional-order systems. According to the definition, a new sufficient condition is established to ensure the considered system is FTB and satisfies the given passive index.
- (iii) Some new sufficient criteria are presented in the form of LMIs, which can be effectively solved by various computational tools.

1.1 Notation

$\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ stand for the n –dimensional vector space and real $(n \times m)$ matrices, respectively. For any matrix $S \in \mathbb{R}^{n \times n}$, $S > 0$ or $S < 0$ means that it is positive-definite or negative-definite matrix respectively. $\lambda_{\max}(S)$ and $\lambda_{\min}(S)$ denote the maximal and the minimal eigenvalue of $S \in \mathbb{R}^{n \times n}$, respectively. For $p \geq 1$, the space of all p –integrable functions on $[0, T]$ denoted by $L_p([0, T])$. The symbol $*$ stands for symmetric block elements in a matrix.

2 Problem Formulation and Preliminaries

We present the concepts of the AB integral ${}^{AB}I_\rho^\alpha f(\rho)$, the AB fractional derivative of Caputo type ${}^{ABC}D_\rho^\alpha f(\rho)$, $\alpha \in (0, 1)$ for a continuously differentiable function $f(\rho)$ such that $\dot{f}(\rho) \in L_1(0, +\infty)$ as follows [5], respectively:

$$\begin{aligned}
{}^{AB}I_{\rho}^{\alpha} f(\rho) &= \frac{1-\alpha}{B(\alpha)} f(\rho) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^{\rho} (\rho-s)^{(\alpha-1)} f(s) ds, \\
{}^{ABC}D_{\rho}^{\alpha} f(\rho) &= \frac{B(\alpha)}{1-\alpha} \int_0^{\rho} \dot{f}(s) E_{\alpha} \left(\frac{-\alpha}{1-\alpha} (\rho-s)^{\alpha} \right) ds,
\end{aligned}$$

where $\Gamma(s) = \int_0^{\infty} e^{-u} u^{s-1} du$, $s > 0$ is the Gamma function, $B(\alpha)$ denotes a real-valued normalization function satisfying $B(\alpha) > 0$, and $B(0) = B(1) = 1$.

We recall some useful lemmas on AB integral and AB fractional derivative of Caputo type of order $\alpha \in (0, 1)$.

Lemma 1 [5] *If $f(\rho)$ is a continuously differentiable function such that $\dot{f}(\rho) \in L_1(0, +\infty)$, then we have*

$${}^{AB}I_{\rho}^{\alpha} \left({}^{ABC}D_{\rho}^{\alpha} f(\rho) \right) = f(\rho) - f(0), \quad \forall \rho \geq 0.$$

Lemma 2 [36] *Let $x(t) \in \mathbb{R}^n$ be a vector of differential function and $Q > 0$. Then, for any $\rho \geq 0$, the following inequality holds*

$${}^{ABC}D_{\rho}^{\alpha} x^T(\rho) Q x(\rho) \leq 2x^T(\rho) Q {}^{ABC}D_{\rho}^{\alpha} x(\rho).$$

Consider the following nonlinear ABC FO systems:

$$\begin{cases}
{}^{ABC}D_{\rho}^{\alpha} x(\rho) = [A + \Delta A(\rho)] x(\rho) \\
\quad + [W + \Delta W(\rho)] \omega(\rho) + f(\rho, x(\rho), \omega(\rho)), \quad \rho \geq 0, \\
z(\rho) = Mx(\rho) + N\omega(\rho), \quad \rho \geq 0, \\
x(0) = x_0 \in \mathbb{R}^n,
\end{cases} \quad (1)$$

where $\alpha \in (0, 1)$, $x(\rho) \in \mathbb{R}^n$ is the state vector, $\omega(\rho) \in \mathbb{R}^p$ is the disturbance input, $z(\rho) \in \mathbb{R}^q$ is the output vector, $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times p}$, $M \in \mathbb{R}^{q \times n}$, $N \in \mathbb{R}^{q \times p}$ are given constant matrices, $\Delta A(\rho) = E_a F_a(\rho) H_a$, $\Delta W(\rho) = E_w F_w(\rho) H_w$, in which E_a , H_a , E_w , H_w are given real constant matrices and $F_a(\rho)$, $F_w(\rho)$ are unknown matrices satisfying $F_a^T(\rho) F_a(\rho) \leq I$, $F_w^T(\rho) F_w(\rho) \leq I$, $\forall \rho \geq 0$. For further development, the following assumptions are needed.

Assumption 1 The nonlinear function $f(\rho, x(\rho), \omega(\rho))$ satisfying the growth Lipschitzian condition

$$\begin{aligned}
&f^T(\rho, x(\rho), \omega(\rho)) f(\rho, x(\rho), \omega(\rho)) \\
&\leq x^T(\rho) G_x^T G_x x(\rho) + \omega^T(\rho) G_w^T G_w \omega(\rho), \quad \forall \rho \geq 0,
\end{aligned}$$

where G_x , G_w are known constant matrices.

Assumption 2 The disturbance input $\omega(\rho) \in \mathbb{R}^p$ is the continuous function satisfying

$$\exists \bar{\omega} > 0 : \sup_{\rho \geq 0} \omega^T(\rho) \omega(\rho) \leq \bar{\omega}.$$

Definition 1 [36] Given positive scalars $T_f, r_1, r_2 (r_1 < r_2), \bar{\omega}$ and $S > 0$. System (1) is said to be robustly finite-time boundedness with respect to (w.r.t) $(r_1, r_2, S, T_f, \bar{\omega})$ if $x^T(0)Sx(0) \leq r_1 \Rightarrow x^T(\rho)Sx(\rho) < r_2, \forall \rho \in [0, T_f]$ and the disturbance input $\omega(\rho) \in \mathbb{R}^p$ satisfying Assumption 2.

Definition 2 System (1) is said to be finite-time passive (FTP) w.r.t $(r_1, r_2, S, T_f, \bar{\omega})$ if two the following conditions are fulfilled:

- (i) System (1) with $z(\rho) \equiv 0$ is robustly FTB w.r.t $(r_1, r_2, S, T_f, \bar{\omega})$.
- (ii) Under the zero initial condition $x(0) = 0, \exists \gamma > 0$ such that $\forall \rho \in [0, T_f]$, we have

$$2^{AB} I_{\rho}^{\alpha} \left(z^T(\rho) \omega(\rho) \right) \geq -\gamma^{AB} I_{\rho}^{\alpha} \left(\omega^T(\rho) \omega(\rho) \right).$$

Remark 1 There are some definitions of fractional-order derivatives in the field of fractional-order calculus. The most widely employed definitions include Grunwald–Letnikov, Riemann–Liouville, and Caputo fractional-order derivatives. The advantages and disadvantages of these fractional-order derivatives have been discussed in [40]. In [3, 19], the authors pointed out that many real-life phenomena cannot be implemented exactly by these fractional-order derivatives due to there exists an artificial singularity at the initial point of the power-law kernel. To overcome these disadvantages, Caputo and Fabrizio [8] proposed a novel fractional differential based upon a non-singular exponential kernel, which eliminated the limitation of singularity in the power-law kernel. However, CF fractional derivative with an exponential kernel is not non-local in nature [18]. Recently, Atangana and Baleanu [3] proposed two novel fractional-differential operators, namely Atangana–Baleanu in Riemann–Liouville sense and Atangana–Baleanu in Caputo sense based upon non-local and non-singular Mittag–Leffler function which itself possesses a strong connection with the fractional differential theory [19]. ABC fractional differential operator based upon the Mittag–Leffler kernel exhibits all the advantages of conventional Caputo fractional differential operator; in addition, the kernel associated with is non-singular. ABC fractional-order systems have received a lot of attention in recent years due to many applications in modeling many real-life phenomena [1, 2, 4, 17, 23, 25, 30, 31, 34, 36–39, 44]. Therefore, we focus on the problem of finite-time passivity for nonlinear fractional-order systems subject to ABC derivative in this paper.

3 Main Results

In order to study the problem of passivity analysis for system (1), we first present a sufficient condition for the FTB of system (1) with $z(t) \equiv 0$.

Theorem 1 System (1) with $z(\rho) \equiv 0$ is robustly FTB w.r.t $(r_1, r_2, S, T_f, \bar{\omega})$ if there exist a matrix $Q > 0$, positive scalars v_1, v_2, v, ϵ satisfying the following conditions

$$\begin{bmatrix} \Xi_{11} & Q & QW & QE_a & QE_w \\ * & -\nu I & 0 & 0 & 0 \\ * & * & \Xi_{33} & 0 & 0 \\ * & * & * & -\nu_1 I & 0 \\ * & * & * & * & -\nu_2 I \end{bmatrix} < 0, \quad (2a)$$

$$r_1 s_2 + \frac{\epsilon(1-\alpha)\bar{\omega}}{B(\alpha)} + \frac{\epsilon\alpha\bar{\omega}}{B(\alpha)\Gamma(\alpha+1)} T_f^\alpha \leq s_1 r_2, \quad (2b)$$

where $\bar{Q} = S^{-\frac{1}{2}} Q S^{-\frac{1}{2}}$, $s_1 = \lambda_{\min}(\bar{Q})$, $s_2 = \lambda_{\max}(\bar{Q})$, $\Xi_{11} = QA + A^T Q + \nu_1 H_a^T H_a + \nu G_x^T G_x$, $\Xi_{33} = -\epsilon I + \nu_2 H_w^T H_w + \nu G_w^T G_w$.

Proof Consider the candidate quadratic function

$$V(\rho) = V(x(\rho)) = x^T(\rho) Q x(\rho).$$

It follows from Lemma 2 that we calculate the α -order AB fractional derivative of $V(\rho)$ as follows:

$$\begin{aligned} {}^{ABC}D_\rho^\alpha V(\rho) &\leq 2x^T(\rho) Q {}^{ABC}D_\rho^\alpha x(\rho) \\ &= x^T(\rho) \left[QA + A^T Q \right] x(\rho) \\ &\quad + 2x^T(\rho) QE_a F_a(\rho) H_a x(\rho) + 2x^T(\rho) QW \omega(\rho) \\ &\quad + 2x^T(\rho) QE_w F_w(\rho) H_w \omega(\rho) + 2x^T(\rho) Qf(\rho, x(\rho), \omega(\rho)). \end{aligned} \quad (3)$$

We get the following estimates from the Cauchy matrix inequality

$$2x^T(\rho) QE_a F_a(\rho) H_a x(\rho) \leq \nu_1^{-1} x^T(\rho) QE_a E_a^T Q x(\rho) + \nu_1 x^T(\rho) H_a^T H_a x(\rho), \quad (4)$$

$$2x^T(\rho) QE_w F_w(\rho) H_w \omega(\rho) \leq \nu_2^{-1} x^T(\rho) QE_w E_w^T Q x(\rho) + \nu_2 \omega^T(\rho) H_w^T H_w \omega(\rho). \quad (5)$$

From Assumption 1, for any positive scalar ν , we get

$$\begin{aligned} &\nu x^T(\rho) G_x^T G_x x(\rho) + \nu \omega^T(\rho) G_w^T G_w \omega(\rho) \\ &\quad - \nu f^T(\rho, x(\rho), \omega(\rho)) f(\rho, x(\rho), \omega(\rho)) \geq 0. \end{aligned} \quad (6)$$

From (3)–(6), we obtain

$${}^{ABC}D_\rho^\alpha V(\rho) \leq \zeta^T(\rho) \Psi \zeta(\rho) + \epsilon \omega^T(\rho) \omega(\rho), \quad (7)$$

where

$$\begin{aligned} \zeta(\rho) &= \begin{bmatrix} x(\rho) \\ f(\rho, x(\rho), \omega(\rho)) \\ \omega(\rho) \end{bmatrix}, \quad \Psi = \begin{bmatrix} \Psi_{11} & Q & QW \\ * & -\nu I & 0 \\ * & * & \Psi_{33} \end{bmatrix}, \\ \Psi_{11} &= QA + A^T Q + \nu_1^{-1} QE_a E_a^T Q + \nu_1 H_a^T H_a + \nu_2^{-1} QE_w E_w^T Q + \nu G_x^T G_x, \end{aligned}$$

$$\Psi_{33} = -\epsilon I + \nu_2 H_w^T H_w + \nu G_w^T G_w.$$

In the light of Schur complement lemma, we can conclude that $\Psi < 0$ is equivalent to (2a). From (2a) and (7), we obtain

$${}^{ABC}D_\rho^\alpha V(\rho) \leq \epsilon \omega^T(\rho) \omega(\rho), \quad \forall \rho \in [0, T_f]. \quad (8)$$

Taking Atangana–Baleanu integral both sides of (8) from 0 to ρ ($0 < \rho < T_f$) and using Lemma 1 together with Assumption 2 gives

$$\begin{aligned} x^T(\rho) Q x(\rho) &\leq x^T(0) Q x(0) + \epsilon \frac{1-\alpha}{B(\alpha)} \omega^T(\rho) \omega(\rho) \\ &\quad + \frac{\epsilon \alpha}{B(\alpha) \Gamma(\alpha)} \int_0^\rho (\rho-s)^{(\alpha-1)} \omega^T(s) \omega(s) ds \\ &\leq x_0^T Q x_0 + \frac{\epsilon(1-\alpha)\bar{\omega}}{B(\alpha)} + \frac{\epsilon \alpha \bar{\omega}}{B(\alpha) \Gamma(\alpha)} \int_0^\rho (\rho-s)^{\alpha-1} ds \\ &\leq x_0^T Q x_0 + \frac{\epsilon(1-\alpha)\bar{\omega}}{B(\alpha)} + \frac{\epsilon \alpha \bar{\omega}}{B(\alpha) \Gamma(\alpha+1)} T_f^\alpha. \end{aligned} \quad (9)$$

Since $\bar{Q} = S^{-\frac{1}{2}} Q S^{-\frac{1}{2}}$, we imply

$$Q = S^{\frac{1}{2}} \bar{Q} S^{\frac{1}{2}}. \quad (10)$$

Using Eq. (10) and some simple computation, we get the following estimates

$$x^T(\rho) Q x(\rho) = x^T(\rho) S^{\frac{1}{2}} \bar{Q} S^{\frac{1}{2}} x(\rho) \geq \lambda_{\min}(\bar{Q}) x^T(\rho) S x(\rho) = s_1 x^T(\rho) S x(\rho), \quad (11)$$

and

$$x_0^T Q x_0 = x_0^T S^{\frac{1}{2}} \bar{Q} S^{\frac{1}{2}} x_0 \leq \lambda_{\max}(\bar{Q}) x_0^T S x_0 = s_2 x_0^T S x_0 \leq r_1 s_2. \quad (12)$$

Combining (9), (11) and (12), we obtain

$$\begin{aligned} s_1 x^T(\rho) S x(\rho) &\leq x^T(\rho) Q x(\rho) = V(t) \leq r_1 s_2 + \frac{\epsilon(1-\alpha)\bar{\omega}}{B(\alpha)} \\ &\quad + \frac{\epsilon \alpha \bar{\omega}}{B(\alpha) \Gamma(\alpha+1)} T_f^\alpha. \end{aligned} \quad (13)$$

Taking (2b) into account, we finally conclude $x^T(\rho) S x(\rho) < r_2$. Thus, system (1) with $z(\rho) \equiv 0$ is robustly FTB w.r.t $(r_1, r_2, S, T_f, \bar{\omega})$. $\square$

Remark 2 By using Atangana–Baleanu fractional-order Gronwall inequality and the Laplace transform of ABC derivative, the authors in [36] investigated the problem of

FTS and FTB for a class of linear ABC fractional-order systems. However, the issue of nonlinear perturbations, i.e., the terms $f(\rho, x(\rho), \omega(\rho))$ and $\Delta A(\rho), \Delta W(\rho)$, did not consider in this work. Theorem 1 has solved the problem for the nonlinear perturbations ABC fractional-order system (1) with zero output vector based on using LMI approach and using some properties of AB calculus. Thus, the model considered in this paper is more general than the results in [36].

We consider the following linear ABC fractional-order system, which was considered in [36]:

$$\begin{cases} {}^{ABC}D_{\rho}^{\alpha}x(\rho) = Ax(\rho) + W\omega(\rho), \rho \geq 0, \\ x(0) = x_0 \in \mathbb{R}^n. \end{cases} \quad (14)$$

From Theorem 1, we immediately have the following corollary.

Corollary 1 System (14) is robustly FTB w.r.t $(r_1, r_2, S, T_f, \bar{\omega})$ if there exist a matrix $Q > 0$, and a positive scalar ϵ satisfying the following conditions

$$\begin{bmatrix} QA + A^T Q & QW \\ W^T Q & -\epsilon I \end{bmatrix} < 0, \quad (15a)$$

$$r_1 s_2 + \frac{\epsilon(1-\alpha)\bar{\omega}}{B(\alpha)} + \frac{\epsilon\alpha\bar{\omega}}{B(\alpha)\Gamma(\alpha+1)} T_f^{\alpha} \leq s_1 r_2, \quad (15b)$$

where $\bar{Q} = S^{-\frac{1}{2}} Q S^{-\frac{1}{2}}$, $s_1 = \lambda_{\min}(\bar{Q})$, $s_2 = \lambda_{\max}(\bar{Q})$.

Next, we will solve the problem of finite-time passivity for system (1).

Theorem 2 For given positive scalars $r_1, r_2 (r_1 < r_2)$, T_f and a matrix $S > 0$. Assume that Assumptions 1 and 2 are fulfilled. System (1) is robustly FTP w.r.t $(r_1, r_2, S, T_f, \bar{\omega})$ if there exist a matrix $Q > 0$, four positive scalars v_1, v_2, v, γ satisfying the following conditions

$$\begin{bmatrix} \Phi_{11} & Q & (QW - M^T) & QE_a & QE_w \\ * & -vI & 0 & 0 & 0 \\ * & * & \Phi_{33} & 0 & 0 \\ * & * & * & -v_1 I & 0 \\ * & * & * & * & -v_2 I \end{bmatrix} < 0, \quad (16a)$$

$$r_1 s_2 + \frac{\gamma(1-\alpha)\bar{\omega}}{B(\alpha)} + \frac{\gamma\alpha\bar{\omega}}{B(\alpha)\Gamma(\alpha+1)} T_f^{\alpha} \leq s_1 r_2, \quad (16b)$$

where $\bar{Q} = S^{-\frac{1}{2}} Q S^{-\frac{1}{2}}$, $s_1 = \lambda_{\min}(\bar{Q})$, $s_2 = \lambda_{\max}(\bar{Q})$, $\Phi_{11} = QA + A^T Q + v_1 H_a^T H_a + v G_x^T G_x$, $\Phi_{33} = -\gamma I + v_2 H_w^T H_w + v G_w^T G_w - (N + N^T)$.

Proof Whenever $z(\rho) \equiv 0$, (15a) and (15b) imply (16a) and (16b). It follows from Theorem 1, system (1) with $z(\rho) \equiv 0$ is robustly finite-time boundedness w.r.t

$(r_1, r_2, S, T_f, \bar{\omega})$. To prove the FTP analysis of the system (1), we choose the quadratic function as used in the proof of Theorem 1. We get the estimates

$${}^{ABC}D_{\rho}^{\alpha}V(\rho) - 2z^T(\rho)\omega(\rho) - \gamma\omega^T(\rho)\omega(\rho) \leq \xi^T(\rho)\mathcal{M}\xi(\rho), \quad (17)$$

where

$$\begin{aligned} \xi(\rho) &= \begin{bmatrix} x(\rho) \\ f(\rho, x(\rho), \omega(\rho)) \\ \omega(\rho) \end{bmatrix}, \\ \mathcal{M} &= \begin{bmatrix} \mathcal{M}_{11} & Q & QW - M^T \\ * & -\nu I & 0 \\ * & * & \mathcal{M}_{33} \end{bmatrix}, \\ \mathcal{M}_{11} &= QA + A^T Q + \nu_1^{-1}QE_aE_a^TQ + \nu_1H_a^TH_a + \nu_2^{-1}QE_wE_w^TQ + \nu G_x^TG_x, \\ \mathcal{M}_{33} &= -\gamma I + \nu_2H_w^TH_w + \nu G_w^TG_w - (N + N^T). \end{aligned}$$

In the light of the Schur complement lemma, we can conclude that $\mathcal{M} < 0$ is equivalent to (16a). From (16a) and (17), we get

$${}^{ABC}D_{\rho}^{\alpha}V(\rho) \leq 2z^T(\rho)\omega(\rho) + \gamma\omega^T(\rho)\omega(\rho), \quad \forall \rho \in [0, T_f]. \quad (18)$$

Taking Atangana–Baleanu integral both sides of (18) from 0 to ρ ($0 < \rho < T_f$) and using Lemma 1 gives

$$V(\rho) - V(0) \leq {}^{AB}I_{\rho}^{\alpha} \left(2z^T(\rho)\omega(\rho) + \gamma\omega^T(\rho)\omega(\rho) \right). \quad (19)$$

Under the zero initial condition, we have $V(0) = x^T(0)Qx(0) = 0$. Combining this fact with $V(\rho) \geq 0$, we get

$${}^{AB}I_{\rho}^{\alpha} \left(2z^T(\rho)\omega(\rho) + \gamma\omega^T(\rho)\omega(\rho) \right) \geq 0, \quad \forall \rho \in [0, T_f].$$

Hence $2 {}^{AB}I_{\rho}^{\alpha} (z^T(\rho)\omega(\rho)) \geq -\gamma {}^{AB}I_{\rho}^{\alpha} (\omega^T(\rho)\omega(\rho))$, $\forall \rho \in [0, T_f]$. According to Definition 2, system (1) is robustly finite-time passive w.r.t $(r_1, r_2, S, T_f, \bar{\omega})$. $\square$

Remark 3 Although there have been many important results on finite-time passivity problems for some kinds of integer-order dynamical systems [10, 32, 45, 46], these results cannot be easily generalized and applied to ABC fractional-order systems since the semigroup property is not valid for Atangana–Baleanu calculus [5]. By employing the theory of finite-time stability and some properties of Atangana–Baleanu calculus, this paper has addressed the problem of the ABC fractional-order nonlinear system for the first time. This is the key contribution of this paper.

Remark 4 Theoretically, the LMI toolbox can solve effectively for any arbitrary dimension LMIs. However, when verifying the feasibility of the obtained LMIs, the increase

in the number of the decision variables will cause high computational complexity and be more time-consuming in actual application, which can be viewed as the disadvantage of the LMI-based approach. However, these problems can be solved in the face of powerful processors. From Theorem 2, the following algorithm is proposed to find the feasible solution of the conditions (16a) and (16b):

- Step 1. Solve the LMI-based condition (16a) to obtain five positive scalars $\gamma, \epsilon, \nu, \nu_1, \nu_2$ and a matrix $Q > 0$.
- Step 2. Calculate $\overline{Q} = S^{-\frac{1}{2}} Q S^{-\frac{1}{2}}, s_1 = \lambda_{\min}(\overline{Q}), s_2 = \lambda_{\max}(\overline{Q})$.
- Step 3. Check the condition (16b) in Theorem 2.

4 Numerical Examples

Example 1 Let us consider the linear Atangana–Baleanu–Caputo fractional-order system (14) with the following parameters

$$\alpha = 0.5, A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}, W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In order to compare with the current existing work in [36], we consider two cases:

- Case 1: Take $r_1 = 1, T_f = 10, S = I, \overline{\omega} = 0.1$ and $B(\alpha) = 1$. Using Corollary 1, the system is robustly finite-time boundedness w.r.t $(1, r_2, I, 10, 0.1)$ for any $r_2 \geq 3.5$. Using Theorem 3 in [36], the minimum value of r_2 is 5.5.
- Case 2: Take $r_1 = 1, r_2 = 5, S = I, \overline{\omega} = 0.2$ and $B(\alpha) = 1$. According to Corollary 1, the system is robustly finite-time boundedness w.r.t $(1, 5, I, T_f, 0.2)$ for any interval finite-time $0 < T_f \leq T_{f \max} = 15$. Using Theorem 3 in [36], the maximum value of $T_{f \max}$ is 0.772.

From the above discussions, our results are more effective with wider applications than those in [36].

Example 2 Consider the system (1) with the following parameters

$$A = \begin{bmatrix} -2 & 0.1 \\ 0 & -2 \end{bmatrix}, E_a = \begin{bmatrix} 0.5 \\ 0.06 \end{bmatrix}, H_a = [0.7 \ 0.8], F_a(\rho) = \sin \rho,$$

$$W = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, E_w = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, H_w = [0.5], F_w(\rho) = \cos \rho,$$

$$M = [2 \ 3], N = [0.5].$$

The nonlinear perturbation term $f(\rho, x(\rho), \omega(\rho))$ satisfying Assumption 1 with $G_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, G_w = [1]$. We choose the disturbance input $\omega(\rho) = \sqrt{0.1} \cos 2\rho \in \mathbb{R}$. It is not hard to verify that the disturbance satisfying Assumption 2 with $\overline{\omega} = 0.1$. Let $r_1 = 1, r_2 = 2, T_f = 10$, the matrix $S = I$ and $B(\alpha) = 1$. Using Remark 4,

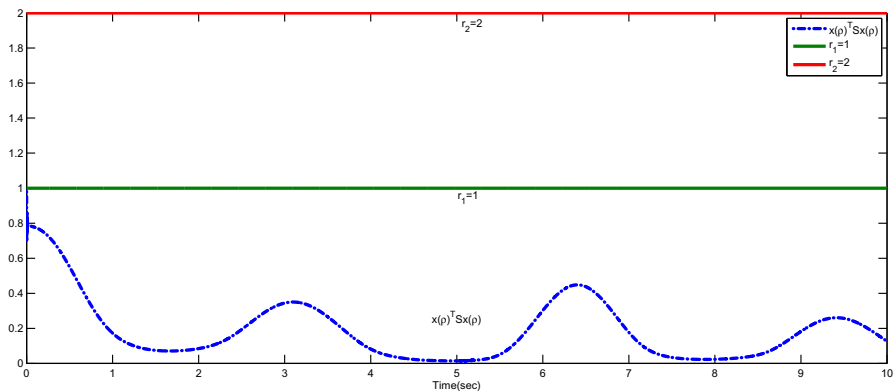


Fig. 1 State response of $x(\rho)^T Sx(\rho)$

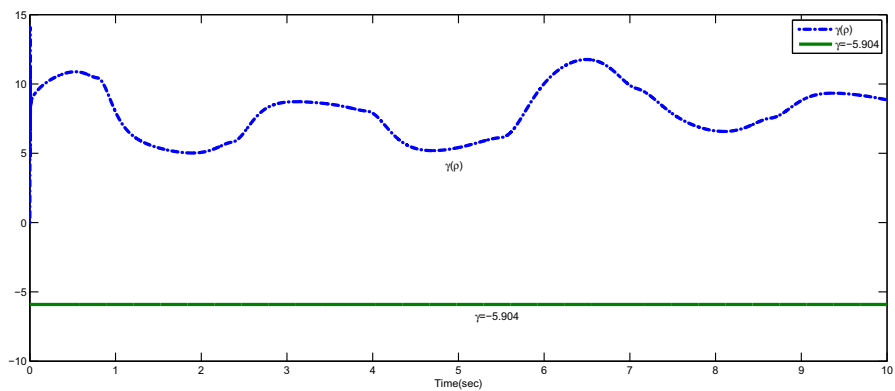


Fig. 2 State response of $\gamma(\rho)$ and $\gamma = -5.9004$.

we have founded the feasible solution of the conditions (16a) and (16b) as follows $\gamma = 5.9004$, $\nu = 3.3398$, $\nu_1 = 1.7493$, $\nu_2 = 8.4628$ and $Q = \begin{bmatrix} 2.5773 & -0.0803 \\ -0.0803 & 2.3702 \end{bmatrix}$. According to Theorem 2, the system is robustly FTP w.r.t $(1, 2, I, 10, 0.1)$.

Simulation results: we consider the initial values $x(0) = (0.7 \ 0.7)^T \in \mathbb{R}^2$. Figure 1 shows the trajectories of $x(\rho)^T Sx(\rho)$ of the system. From Fig. 1, it is easy to see that the system is robustly FTP w.r.t $(1, 2, I, 10, 0.1)$. To check the passivity performance, we define a passivity performance function as $\gamma(\rho) := \frac{2^{AB} I_\rho^\alpha(z^T(\rho)\omega(\rho))}{AB I_\rho^\alpha(\omega^T(\rho)\omega(\rho))}$, $\rho \in [0, 10]$ and $x_0 = 0$. Figure 2 indicates that the passivity performance function $\gamma(\rho)$ is always greater than $-\gamma = -5.9004$, which verifies that the condition (ii) in Definition 2 always holds.

5 Conclusion

This paper has addressed the problem of robustly finite-time passivity of nonlinear ABC fractional-order systems. The contribution of this paper is to introduce the definition of finite-time passivity for ABC fractional-order systems and to derive a new sufficient condition that ensure the considered system is FTB and satisfies the given passive index. Two numerical examples are presented to verify the correctness of the obtained results. In future work, we will focus on the problem for nonlinear ABC fractional-order systems with various types of time delays. Also, finding a new finite-time passivity condition for other kinds fractional-order systems, including singular systems [11, 12], delayed fractional memristor neural networks [22], will be handled as future extension works.

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Data Availability All data generated or analyzed during this study are included in this article.

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